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Note

Further results about large sets of disjoint Mendelsohn triple systems

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Abstract

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In this note, a construction of the large sets of pairwise disjoint Mendelsohn triple systems of order $72k+6$, where $k>1$ and $k\equiv 1$ or 2 (mod 3), is given.

Let X be a set of v elements ($v\geqslant 3$). A *cyclic triple* from X is a collection of three pairs (x,y), (y,z) and (z,x), where x,y and z are distinct elements of X. The cyclic triple is denoted by $\langle x,y,z\rangle$ or $\langle y,z,x\rangle$ or $\langle z,x,y\rangle$. A *Mendelsohn triple system* on X is a pair (X,B) where B is a collection of cyclic triples from X such that each ordered pair of distinct elements of X is covered by a unique cyclic triple from B. The system on $X(|X|=v)$ is denoted by MTS(v). Mendelsohn [1] proved that the spectrum for MTS(v)s is the set of all $v\equiv 0$ or 1 (mod 3) and $v\neq 6$.

A *large set of pairwise disjoint* MTS(v)*s* is a collection of $v-2$ pairwise disjoint MTS(v)s. It is denoted by LMTS(v). Up to now, all known results about LMTS(v) are:

(R1) There exists an LMTS(v) for $v\equiv 1,3$ (mod 6) [2];

(R2) If there exists an LMTS(v), then there exists an LMTS($3v$) [3];

(R3) If there exists an LMTS($v+1$) and $v\geqslant 3$, then there exists an LMTS($3v+1$) [3];

(R4) For $p\equiv 1$ or 5 (mod 6), if there exists an LMTS($q+2$), then there exists an LMTS($pq+2$) [2, 4];

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(R5) If there exists an GLS$(2+m)$, then there exists an LMTS$(2+2m)$ [5] (The notion of GLS is due to Lu [11].)

(R6) There exists an LMTS(2^n+2) for $n \geqslant 3$ [6].

All that remains for a complete solution of the problem is $v \equiv 6$ or 22 (mod 72), $v > 6$. In this note we will construct some LMTS(v)s for unknown orders v.

The quasi-symmetric LMTS

In [2], we introduced a special kind of LMTS, namely the symmetric LMTS $\{(X, B_i)\}_i$ defined as follows. There exist $a \neq b \in X$ such that the following conditions hold for any i and $x, y \notin \{a, b\}$:

(1) $\langle a, b, x\rangle \in B_i$ if and only if $\langle b, a, x\rangle \in B_i$;

(2) $\langle a, x, y\rangle \in B_i$ if and only if $\langle b, y, x\rangle \in B_i$.

For the convenience, call the special elements a and b the *nuclei*. In this paper, another kind of LMTS is repeatedly used. This is the so-called quasi-symmetric LMTS, which only satisfies the condition (1) above-mentioned.

By known constructions we can get some examples of quasi-symmetric LMTS.

Lemma 1. (a) *There exists a quasi-symmetric* LMTS$(n+2)$ *for* $n \equiv 1$ *and* 5 (mod 6).

(b) *There exists a quasi-symmetric* LMTS (2^n+2) *for any positive integer* $n \geqslant 3$.

(c) *For* $q \geqslant 3$, *if there exists a symmetric* LMTS$(p+2)$ *and a quasi-symmetric* LMTS$(q+2)$ *then there exists a quasi-symmetric* LMTS$(pq+2)$.

(d) *For odd* $v \geqslant 3$, *if there exists a quasi-symmetric* LMTS(v) (LMTS$(v+1)$, *resp.*) *then there exists a quasi-symmetric* LMTS$(3v)$ (LMTS$(3v+1)$, *resp.*).

Proof. (a), (b) These are trivial by [2] and [6].

(c) By Theorem 6 in [2], since LMTS$(q+2) = \{(\{a, b\} \cup Q, b_j),\ j \in Q\}$ is quasi-symmetric, so $\langle a, b, j\rangle$, $\langle b, a, j\rangle \in b_j$ implies $\langle a, b, (i, j)\rangle$, $\langle b, a, (i, j)\rangle \in T_{ij}$ for any $i \in I_p$, $j \in Q$ (see part (4) of Theorem 6 in [2]).

(d) Firstly, for odd $v \geqslant 3$ there exists a symmetric idempotent Latin square Q of order v having v disjoint transversals (see Remark (1)). Denote the corresponding quasigroup by $(Q, \cdot)$. In the construction of Theorem 2.1 in [3], if the nuclei of the known quasi-symmetric LMTS$(v) = \{(Q, q_k)$: $k = 1, 2, \ldots, v-2\}$ consists of $a, b \in Q$ (i.e. $\langle a, b, k\rangle$, $\langle b, a, k\rangle \in q_k$), then we have:

$$\langle (a,1), (b,1), ((a\cdot b)\alpha_k, 2)\rangle,\ \langle (b,1), (a,1), ((b\cdot a)\alpha_k, 2)\rangle \in t_k \qquad \text{(part(1) (ii))};$$

$$\langle (a,1), (b,1), ((a\cdot b)\alpha_k^*, 3)\rangle,\ \langle (b,1), (a,1), ((b\cdot a)\alpha_k^*, 3)\rangle \in t_k^* \qquad \text{(part(2) (ii))};$$

$$\langle (a,1), (b,1), (k,1)\rangle,\ \langle (b,1), (a,1), (k,1)\rangle \in d_k \qquad \text{(part(3) (i))}.$$

So, we can regard elements $(a, 1)$, $(b, 1) \in Q \times \{1, 2, 3\}$ as the nuclei of the LMTS$(3v) = \{(Q \times \{1, 2, 3\}, t_k);\ \ k = 1, 2, \ldots, v\} \cup \{(Q \times \{1, 2, 3\}, t_k^*);\ \ k = 1, 2, \ldots, v\} \cup \{(Q \times \{1, 2, 3\}, d_k);\ k = 1, 2, \ldots, v-2\}$. $\square$

Remarks. (1) A construction of a symmetric idempotent Latin square Q of order v having v disjoint transversals is $Q=(a_{ij})_0^{v-1}$, where $a_{ij}=2^{-1}(i+j) \pmod v$, and its orthogonal mate is $P=(b_{ij})_0^{v-1}$, where $b_{ij}=2i+j \pmod v$. Here v is odd and $v \geqslant 3$.

(2) We point out that, in construction 3, if LMTS$(q+2)$ is symmetric then LMTS$(pq+2)$ is as well. And for $n \equiv 1$ or 5 (mod 6) the construction LMTS$(n+2)$ provided in [2] is symmetric.

A recursive construction

An orthogonal OA$(2,k,v)$ is a subset L of S^k, S a v-set, such that for any 2-subset $\{i_1,i_2\}$ of $\{1,2,\dots,k\}$ and any pair x_1,x_2 of (not necessarily distinct) elements of S, there is exactly one element $(y_1,\dots,y_k)$ of L with $y_{i_1}=x_1$ and $y_{i_2}=x_2$.

An LD(v) is a collection consisting of $v+2$ sets, $\{L^1, L^2, L_x;\ x \in X\}$, where:

(I) X is a v-set;

(II) L^1 and L^2 are two OA$(2,4,v)$ on X;

(III) there is an element c_0 of X such that $(x,x,x,c_0) \in L^1 \cap L^2$ for all $x \in X$;

(IV) for each $x \in X$, L_x is an OA$(2,3,n-1)$ on $X \setminus \{x\}$;

(V) any $(x_1,x_2,x_3) \in X^3$ is either contained in an L_x, $x \in X$, or in $\bar{L}^1$ or $\bar{L}^2$, where $\bar{L}^j=\{(x_1,x_2,x_3);$ there is an $x \in X$ with $(x_1,x_2,x_3,x) \in L^j\}$, $j=1,2$.

The notion of LD(v) is due to Lu [7]. By his research and that of Teirlinck [8], we have the following conclusion about the existence of LD(v).

Lemma 2. *For positive integers $v \geqslant 4$, $v \neq 6$, there exists an* LD(v) *except possibly for* $v=9$, 10, 14, 18, 23, 26, 27, 30, 38, 42, 46, 62, 74, 86, 90 *and* 114 [9].

Theorem 1. *If an* LD(n) *and a quasi-symmetric* LMTS$(n+2)$ *both exist, then an* LMTS$(3n)$ *exists.*

Construction. Let $I_n=\{0,1,\dots,n-1\}$, $a,b \notin I_n$ and $Z_3=\{0,1,2\}$. Given an LD$(n)=\{L^1,L^2,L_x;\ x \in I_n\}$ on I_n (the condition (III) is fulfilled with $c_0=0$) and an LMTS$(n+2)=\{(a,b) \cup I_n, \tilde{b}_x);\ x \in I_n\}$ with $\langle a,b,x\rangle, \langle b,a,x\rangle \in \tilde{b}_x$.

Define n systems B_x $(x \in I_n)$ of cyclic triples on $I_n \times Z_3$ as follows, where the element $(u,k) \in I_n \times Z_3$ is denoted by u_k:

(A1) $\langle x_0,x_1,x_2\rangle$, $\langle x_0,x_2,x_1\rangle$;

(A2) $\langle \alpha_0,\beta_1,\gamma_2\rangle$, $\langle \alpha_0,\gamma_2,\beta_1\rangle$ provided $(\alpha,\beta,\gamma) \in L_x$;

(A3) $\langle u_k,v_k,w_k\rangle$ provided $\langle u,v,w\rangle \in \tilde{b}_x$, $u,v,w \in I_n$ and $k \in Z_3$;

(A4) $\langle x_k,y_{k+1},z_{k+1}\rangle$ provided $\langle a,y,z\rangle \in \tilde{b}_x$, $y,z \in I_n$ and $k \in Z_3$;

(A5) $\langle x_k,s_{k-1},t_{k-1}\rangle$ provided $\langle b,s,t\rangle \in \tilde{b}_x$, $s,t \in I_n$ and $k \in Z_3$.

Define $2(n-1)$ systems C_{ij} $(i \in I_n \setminus \{0\}, j \in \{1,2\})$ of cyclic triples on $I_n \times Z_3$ as follows:

(B1) $\langle x_0,y_1,z_2\rangle$, $\langle x_0,z_2,y_1\rangle$ provided $(x,y,z,i) \in L^j$;

(B2) if $(x, y, z, i)\in L^j$ then the following cyclic triples belong to C_{ij}:

Case $j=1$:

$$\langle x_0, u_1, v_1\rangle \text{ provided } \langle a, u, v\rangle\in\tilde{b}_y;$$

$$\langle y_1, u_2, v_2\rangle \text{ provided } \langle a, u, v\rangle\in\tilde{b}_z;$$

$$\langle z_2, u_0, v_0\rangle \text{ provided } \langle a, u, v\rangle\in\tilde{b}_x.$$

Case $j=2$:

$$\langle x_0, s_2, t_2\rangle \text{ provided } \langle b, s, t\rangle\in\tilde{b}_z;$$

$$\langle y_1, s_0, t_0\rangle \text{ provided } \langle b, s, t\rangle\in\tilde{b}_x;$$

$$\langle z_2, s_1, t_1\rangle \text{ provided } \langle b, s, t\rangle\in\tilde{b}_y.$$

Then, $\{(I_n\times Z_3, B_x);\ x\in I_n\}\cup\{(I_n\times Z_3, C_{ij});\ i\in I_n\backslash\{0\}, j=1,2\}$ is an LMTS$(3n)$.

Proof. (a) Each B_x is an MTS$(3n)$.

Direct calculation shows that B_x contains

$$2+2(n-1)^2+3(n-1)(n-2)/3+3(n-1)+3(n-1)=3n(3n-1)/3$$

cyclic triples, just the number we expected. So we only need to show that every ordered pair P of distinct elements of the set $I_n\times Z_3$ is contained in some cyclic triple of B_x. All the possibilities are exhausted as follows:

(1) $p=(x_i, y_j)$ $(i\neq j\in Z_3)$ is contained in (A1);

(2) $p=(x_i, u_i)$ or (u_i, x_i) $(u\in I_n\backslash\{x\},\ i\in Z_3)$ is contained in (A3);

(3) $p=(x_i, u_j)$ $(u\in I_n\backslash\{x\},\ i\neq j\in Z_3)$ is contained in (A4) or (A5) if $j\equiv i+1$ or $j\equiv i-1$ (mod 3), and similarly for $p=(u_j, x_i)$;

(4) $p=(u_i, v_j)$ $(u, v\in I_n\backslash\{x\},\ i\neq j\in Z_3)$ is contained in (A2);

(5) $p=(u_i, v_i)$ $(u\neq v\in I_n\backslash\{x\},\ i\in Z_3)$. There exists $w\in\{a, b\}\cup I_n$ such that $\langle u, v, w\rangle\in\tilde{b}_x$. If $w=a$ or b then P is contained in (A4) or (A5), otherwise in (A3).

(b) Each C_{ij} is an MTS$(3n)$.

C_{ij} contains $2n+3n(n-1)=3n(3n-1)/3$ cyclic triples. All ordered pairs P of the set $I_n\times Z_3$ occur as follows.

(1) The pair $p=(u_k, v_k)$ $(u\neq v\in I_n,\ k\in Z_3)$ is contained in (B2). For example, consider the case $k=0$ and $j=1$. There exists $x, z\in I_n$, such that $\langle a, u, v\rangle\in\tilde{b}_x$ and $(x, *, z, i)\in L^1$. Then we have $\langle z_2, u_0, v_0\rangle\in C_{i1}$. The remaining cases are similar.

(2) The pair $p=(u_t, u_k)$ $(u\in I_n,\ t\neq k\in Z_3)$ is contained in (B2). For example, consider the case $t=0$, $k=1$ and $j=2$ (similarly for other case of t, k and j) There exists $y\in I_n\backslash\{u\}$ such that $(y, u, *, i)\in L^2$. Furthermore, there exists $v\in I_n\backslash\{u, y\}$ such that $\langle b, v, u\rangle\in\tilde{b}_y$. Then, we have $\langle u_1, v_0, u_0\rangle\in C_{i2}$.

(3) The pair $p=(u_t, v_k)$ $(u\neq v\in I_n,\ t\neq k\in Z_3)$. We only consider the case $t=2$, $k=1$ and $j=1$ (other cases are similar). There exists $z\in I_n\backslash\{v\}$ such that $(*, v, z, i)\in L^1$. If $z=u$, then P is contained on $\langle *_0, u_2, v_1\rangle$ of part (B1). If $z\neq u$, then there exists $w\in I_n\backslash\{z, u\}$ such that $\langle a, w, u\rangle\in\tilde{b}_z$, so P is contained in $\langle v_1, w_2, u_2\rangle$ of part (B2).

(c) $\{B_x; x\in I_n\}\cup\{C_{ij};\ i\in I_n\backslash\{0\}, j=1,2\}$ is an LMTS$(3n)$.

We only need to show that every cyclic triple T from $I_n \times Z_3$ is contained in some B_x or C_{ij} above. All the possibilities are exhausted as follows:

(1) $T=\langle u_k, v_k, w_k\rangle$ $(u\neq v\neq w\neq u\in I_n,\ k\in Z_3)$ is contained in (A3) of B_x if $\langle u,v,w\rangle\in\tilde{b}_x$;

(2) $T=\langle u_0, v_1, w_2\rangle$ or $\langle u_0, w_2, v_1\rangle$ $(u,v,w\in I_n)$: if $u=v=w$ then T is contained in (A1) of B_u; in other cases, by the properties (III) and (V) of LD(n), $(u,v,w)\in L_x$ or $(u,v,w,i\rangle\in L^j$, so T is contained in (A2) of B_x or in (B1) of C_{ij}:

(3) $T=\langle u_t, v_k, w_k\rangle$ $(u,v,w\in I_n,\ t\neq k\in Z_3,\ v\neq w)$ leads to two cases.

Case 1: $k\equiv t+1 \pmod 3$, as for example $t=0$, $k=1$. There exists $x\in I_n\backslash\{v,w\}$ such that $\langle a,v,w\rangle\in\tilde{b}_x$. If $x=u$, then T is contained in (A4) of B_x, or else in (B2) of C_{i1} provided $(u,x,*,i)\in L^1$. The other cases are similar.

Case 2: $k\equiv t-1 \pmod 3$, as for example $t=1$, $k=0$. There exists $x\in I_n\backslash\{v,w\}$ such that $\langle b,v,w\rangle\in\tilde{b}_x$. If $x=u$, then T is contained in (A5) of B_x, else in (B2) of C_{i2} provided $(x,u,*,i)\in L^2$. The other cases are similar.

The proof is completed. □

The main results

Theorem 2. *There exists an* LMTS$(72k+6)$ *for* $k\equiv 1$ *or* $2 \pmod 3$ *and* $k>1$.

Proof. For $k\not\equiv 0 \pmod 3$, let $k=2^t s$ where $t\geqslant 0$ and $2\nmid s$, then $s\equiv 1$ or $5 \pmod 6$. By (a) and (b) of Lemma 1, there exists a symmetric LMTS$(s+2)$ and a quasi-symmetric LMTS$(2^{t+3}+2)$, so there exists a quasi-symmetric LMTS$(2^{t+3}s+2)=$ LMTS$(8k+2)$ by (c) Lemma 1. Furthermore, by (d) of Lemma 1, there exists a quasi-symmetric LMTS$(3(8k+1)+1)=$ LMTS$(24k+4)$. Finally, since there exists an LD$(24k+2)$ for $k\not\equiv 0 \pmod 3$ and $k>1$ by Lemma 2, so there exists an LMTS$(3(24k+2))=$ LMTS$(72k+6)$ by Theorem 1. □

So, by this theorem, the outstanding cases for LMTS(v) are only $v=78=4\cdot 19+2$, $v=72(3k)+6=4(54k+1)+2$ $(k\geqslant 1)$ and $v=72k+22=4(18k+5)+2$ $(k\geqslant 0)$.

Lemma 3. *For* $t\equiv 1$ *or* $5 \pmod 6$ *and* $t>1$, *if* $t=t_1t_2$ $(t_1>1,\ t_2>1)$ *with* $t_1\not\equiv 1 \pmod{54}$, $t_1\not\equiv 5 \pmod{18}$ *and* $t_1\neq 19$ *then there exists an* LMTS$(4t+2)$.

Proof. By the conclusion mentioned before the lemma there exists an LMTS$(4t_1+2)$. Furthermore, since $t_2\equiv 1$ or $5 \pmod 6$, also there exists an LMTS$(4t_1t_2+2)$ by (R4). □

Now, let us consider all orders $4t+2$ with $t\equiv 5 \pmod{18}$ or $t\equiv 1 \pmod{54}$. If t is a composite number, let $t=t_1t_2$ $(t_1>1,\ t_2>1)$. It is easy to verify the following.

(A) For $t\equiv 5 \pmod{18}$, the three possibilities are (1) $t_1\equiv 1$, $t_2\equiv 5 \pmod{18}$, (2) $t_1\equiv 7$, $t_2\equiv 11 \pmod{18}$ and (3) $t_1\equiv 13$, $t_2\equiv 17 \pmod{18}$. For case (2) and (3), by Lemma 3, there exists an LMTS$(4t+2)$. For case (1), after another application of Lemma 3, the only remaining cases are $t_1\equiv 1 \pmod{54}$ or $t_1=19$.

(B) For $t \equiv 1 \pmod{54}$, the four possibilities are (1) $t_1 \equiv t_2 \equiv 1 \pmod{18}$, (2) $t_1 \equiv 5$, $t_2 \equiv 11 \pmod{18}$, (3) $t_1 \equiv 7, t_2 \equiv 13 \pmod{18}$, (4) $t_1 \equiv t_2 \equiv 17 \pmod{18}$. By Lemma 3, the only remaining cases are $t_i \equiv 1 \pmod{54}$ or $t_i = 19$ of Case (1), where $i = 1$ or 2.

Thus, we can get the following.

Theorem 3. *For* $v \equiv 0$ *or* $1 \pmod 3$ *and* $v \neq 6$, *there exists an* LMTS(v) *except possibly for*

$$v = 4q^m 19^n \prod_{i=1}^{t} p_i + 2,$$

where the prime $q \equiv 5 \pmod{18}$, *the primes* $p_i \equiv 1 \pmod{54}$ *and the integers* $n, t \geqslant 0$, $m = 0$, 1 (*but* $m^2 + n^2 + t^2 \neq 0$).

Finally, we point out that all the constructions in (R1)–(R6) together with Theorem 3 reduce the possible exceptions $\leqslant 500$ to $v = 22$, 78, 94, 166, 238, 382, 438 and 454.

References

[1] N.S. Mendelsohn, A natural generalization of Steiner triple systems, Comput. Number Theory (Academic Press, New York, 1971) 323–338.

[2] Q. Kang and Y. Chang, Symmetric Mendelsohn triple systems and large sets of disjoint Mendelsohn triple systems, in: Combinatorial Designs and Applications, Lecture Notes in Pure and Applied Math., Vol. 126 (1990) 69–78.

[3] C.C. Lindner, On the number of disjoint Mendelsohn triple systems, J. Combin. Theory Ser. A. 30 (1981) 326–330.

[4] L. Teirlinck and C.C. Lindner, The construction of large sets of idempotent quasigroups, European J. Combin. 9 (1988) 83–89.

[5] L. Wu, On large sets of disjoint Mendelsohn triple systems, J. Shuzhou Univ., Vol. 3 No. 2 (1987) 1–4.

[6] Q. Kang, The construction of the large sets of disjoint Mendelsohn triple system of order $2^n + 2$, Chinese Science Bulletin 34 (1989) 1041–1044.

[7] J.X. Lu, On large sets of disjoint Steiner triple systems III and V, J. Combin. Theory Ser. A 34 (1983) 156–182 and 37 (1984) 164–188.

[8] L. Teirlinck, A completion of Lu's determination of the spectrum for large sets of disjoint Steiner triple systems, preprint.

[9] Y. Chang, Q. Kang and J. Lei, The existence of the designs LD and LD*, Chinese Science Bulletiin 16 (1990) 1275.

[10] Q. Kang, Large sets of block designs, Ph.D. Thesis, Eindhoven University of Techn., 1989.

[11] J.X. Lu, On large sets of disjoint Steiner triple systems I, J. Combin. Theory Ser. A 34 (1983) 140–146.